

Vector Basics

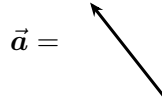
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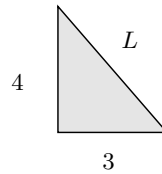
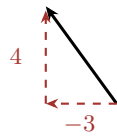
1 Vector Basics

1.1 Magnitude and direction

If $\vec{a} = (-3, 4)$, then



The *magnitude* of $\vec{a}$ is its length, written $|\vec{a}|$.

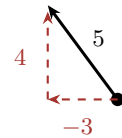


$$\begin{aligned} 3^2 + 4^2 &= L^2, \\ 9 + 16 &= L^2, \\ 25 &= L^2, \\ 5 &= L. \end{aligned}$$

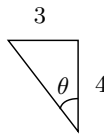
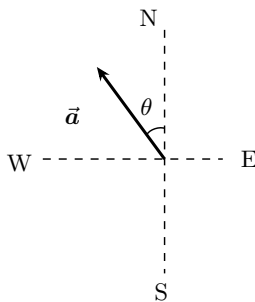
Using Pythagoras, or directly using the components,

$$\begin{aligned} |\vec{a}| &= \sqrt{a_x^2 + a_y^2} \\ &= \sqrt{(-3)^2 + 4^2} \\ &= 5. \end{aligned}$$

So $\vec{a}$ has a magnitude of 5.



The vector $\vec{a}$ also has a direction. To find θ :



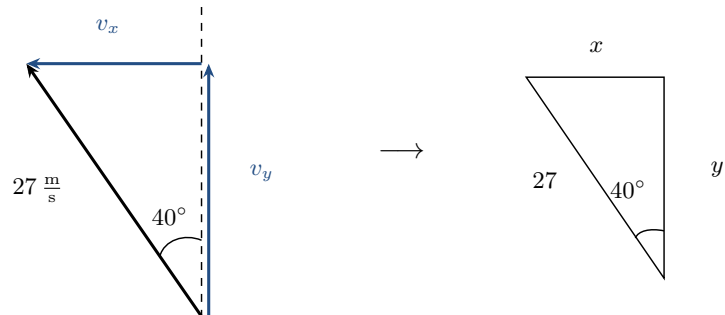
$$\begin{aligned} \tan \theta &= \frac{3}{4}, \\ \theta &= \tan^{-1}\left(\frac{3}{4}\right), \\ \theta &\simeq 36.9^\circ. \end{aligned}$$

Thus $\vec{a} = (-3, 4)$ has magnitude 5 and direction $[36.9^\circ \text{ W of N}]$. Its components are $a_x = -3$ and $a_y = 4$.

1.2 Finding components

Example. $\vec{v} = 27 \frac{\text{m}}{\text{s}} [40^\circ \text{ W of N}]$.

Simplifying to a right triangle gives:



$$\sin 40^\circ = \frac{x}{27},$$

$$\cos 40^\circ = \frac{y}{27},$$

$$x = 27 \sin 40^\circ \simeq 17.4 \frac{\text{m}}{\text{s}}, \quad y = 27 \cos 40^\circ \simeq 20.7 \frac{\text{m}}{\text{s}}.$$

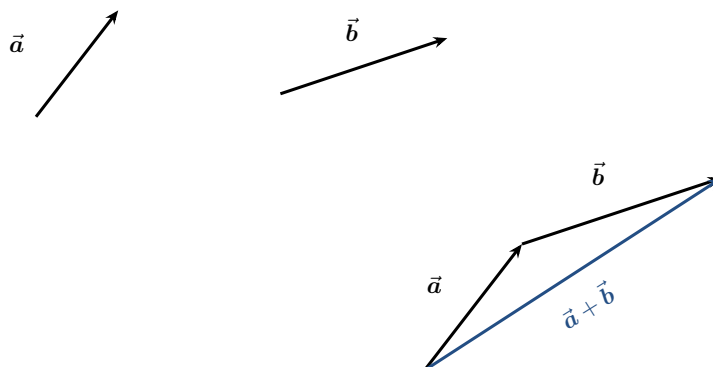
So

$$\vec{v} = \left(\underbrace{-17.4 \frac{\text{m}}{\text{s}}}_{\text{negative}}, \underbrace{20.7 \frac{\text{m}}{\text{s}}}_{\text{positive}} \right),$$

since $\vec{v}$ is going west and north.

2 Adding Vectors

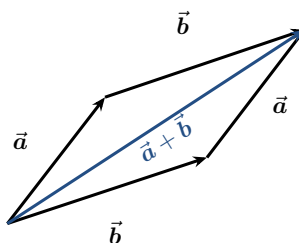
2.1 Tip-to-tail addition



$\vec{a} + \vec{b}$ is also called the *resultant*.

We can see that

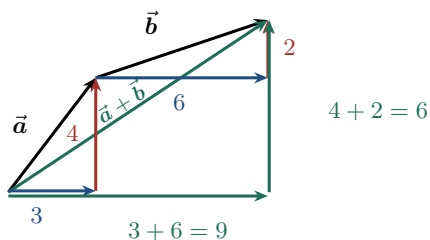
$$\vec{a} + \vec{b} = \vec{b} + \vec{a}.$$



2.2 Using components



For $\vec{a} = (3, 4)$ and $\vec{b} = (6, 2)$,



$$\begin{aligned}\vec{a} + \vec{b} &= (3, 4) + (6, 2) \\ &= (3 + 6, 4 + 2) \\ &= (9, 6).\end{aligned}$$

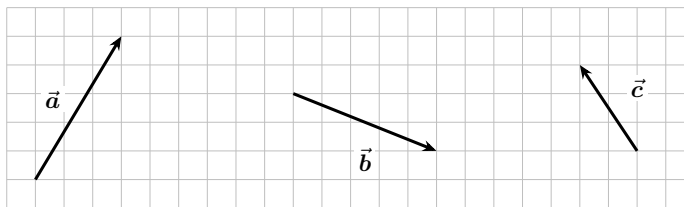
When vectors are written in components, we can add the x -components together and the y -components together.

Example. $\vec{a} = (-7, 3)$, $\vec{b} = (2, -10)$, and $\vec{c} = (5, 8)$.

$$\begin{aligned}\vec{b} + \vec{c} + \vec{a} &= (2, -10) + (5, 8) + (-7, 3) \\ &= (2 + 5 - 7, -10 + 8 + 3) \\ &= (0, 1).\end{aligned}$$

2.3 Exercises

1. Add the following vectors graphically, using the tip-to-tail method on graph paper. Each grid square represents one unit.

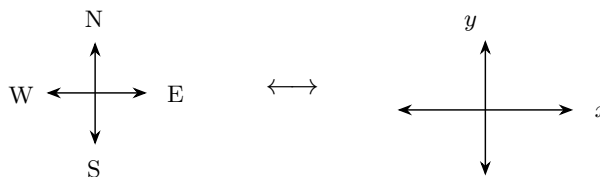


Draw the resultant (sum) for both

$$\vec{a} + \vec{b} + \vec{c} \quad \text{and} \quad \vec{b} + \vec{c} + \vec{a}.$$

What do you notice?

2. The compass directions are equivalent to the coordinate directions:



- a) i) Draw $\vec{a} = 337 \text{ m } [72^\circ \text{ W of N}]$.
 ii) Find a_x and a_y .
 b) i) Draw $\vec{b} = 10 \frac{\text{m}}{\text{s}} [30^\circ \text{ S of E}]$.
 ii) Find b_x and b_y .
3. Find the magnitude and direction of each vector:

a) $\vec{a} = (73.2, -11)$.

b) $\vec{b} = (-43, -20)$.

4. Given

$$\left. \begin{array}{l} \vec{a} = 17 \text{ m } [30^\circ \text{ N of E}], \\ \vec{b} = 33 \text{ m } [80^\circ \text{ W of S}], \\ \vec{c} = 100 \text{ m } [45^\circ \text{ N of W}], \end{array} \right\} \begin{array}{l} \text{polar form} \\ \text{(magnitude and direction)} \end{array}$$

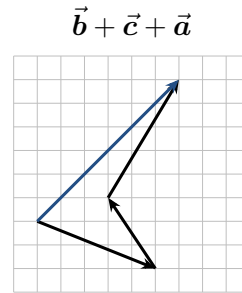
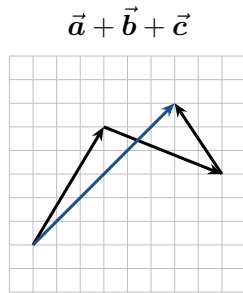
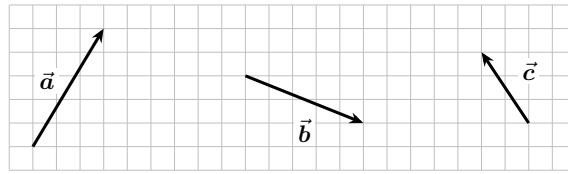
In component form, for comparison:

$$\underbrace{\begin{array}{l} \vec{u} = (71, -5) \quad \text{or} \quad u_x = 71, \quad u_y = -5, \\ \vec{v} = (-23, 8) \quad \quad \quad v_x = -23, \quad v_y = 8. \end{array}}_{\text{component form}}$$

- a) Find $\vec{a} + \vec{b}$ by converting to components first, then give the final answer in polar form.
 b) Find $\vec{b} + \vec{c}$, giving the answer in polar form.
 c) Find $\vec{a} + \vec{b} + \vec{c}$, giving the answer in polar form.

2.4 Answers

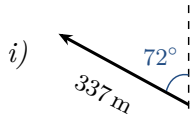
1.



$$\vec{a} + \vec{b} + \vec{c} = \vec{b} + \vec{c} + \vec{a}.$$

Both are equal to (6, 6). Order does not matter when adding, but the vectors must still be arranged tip-to-tail.

2. a)

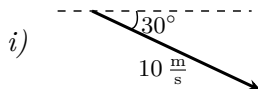


ii)

$$\begin{aligned} a_x &= -337 \sin 72^\circ, \\ a_y &= 337 \cos 72^\circ. \end{aligned}$$

$$\vec{a} \simeq (-320.5 \text{ m}, 104.1 \text{ m}).$$

b)



ii)

$$\begin{aligned} b_x &= 10 \cos 30^\circ = 8.66 \frac{\text{m}}{\text{s}}, \\ b_y &= -10 \sin 30^\circ = -5 \frac{\text{m}}{\text{s}}. \end{aligned}$$

$$\vec{b} = \left(8.66 \frac{\text{m}}{\text{s}}, -5 \frac{\text{m}}{\text{s}} \right).$$

3. a) $|\vec{a}| = 74.02$; direction: 8.55° S of E.b) $|\vec{b}| = 47.42$; direction: 24.94° S of W.4. a) $\vec{a} + \vec{b} = 17.99 \text{ m}$ [8.86° N of W].b) $\vec{b} + \vec{c} = 121.96 \text{ m}$ [32.2° N of W].c) $\vec{a} + \vec{b} + \vec{c} = 115.02 \text{ m}$ [39.7° N of W].

3 Scalar Multiplication

3.1 Multiplying by a scalar

We can add vectors, and we can also multiply them by a number, called a *scalar*.

If $\vec{v} = (3, 4)$, what is $5\vec{v}$? Recall that

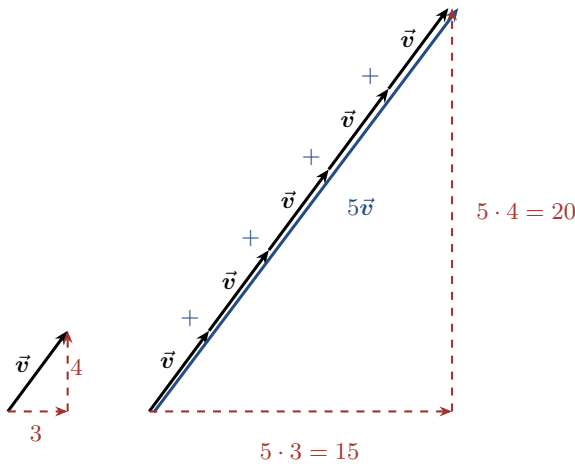
$$5 \cdot 3 = 3 + 3 + 3 + 3 + 3 = 15,$$

so

$$5x = x + x + x + x + x,$$

and therefore

$$5\vec{v} = \vec{v} + \vec{v} + \vec{v} + \vec{v} + \vec{v}.$$



We can see that $5\vec{v}$ has the same direction as $\vec{v}$, but is five times as long. Its components have also been multiplied by 5. Since $\vec{v} = (3, 4)$,

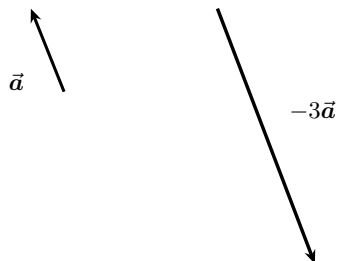
$$\begin{aligned} 5\vec{v} &= 5(3, 4) \\ &= (5 \cdot 3, 5 \cdot 4) \\ &= (15, 20). \end{aligned}$$

3.2 Negative scalars

Example. Calculate $-3\vec{a}$ where $\vec{a} = (-2, 5)$.

$$-3\vec{a} = -3(-2, 5) = (6, -15).$$

Graphically:



The vector is three times as long, with its direction reversed, since it is multiplied by -3 .

Example. If $\vec{v} = (-3, -4)$, then

$$-\vec{v} = (-1)\vec{v} = (-1)(-3, -4) = (3, 4).$$



Multiplying a vector by a negative scalar reverses its direction.

Example. If $\vec{v} = 30 \frac{\text{m}}{\text{s}}$ [28° E of N], then

$$5\vec{v} = 150 \frac{\text{m}}{\text{s}} \text{ [28° E of N]}.$$

The direction does not change; the vector is five times as long.

$$-2\vec{v} = 60 \frac{\text{m}}{\text{s}} \text{ [28° W of S]}.$$

The vector is twice as long and points in the opposite direction.

3.3 Dividing by a scalar

Example. Let $\vec{p} = (12, -60)$. Calculate $\frac{\vec{p}}{3}$.

$$\begin{aligned} \frac{\vec{p}}{3} &= \frac{1}{3}\vec{p} = \frac{1}{3}(12, -60) \\ &= \left(\left(\frac{1}{3} \right) (12), \left(\frac{1}{3} \right) (-60) \right) \\ &= (4, -20). \end{aligned}$$

In other words,

$$\begin{aligned} \frac{\vec{p}}{3} &= \left(\frac{12}{3}, \frac{-60}{3} \right) \\ &= (4, -20). \end{aligned}$$

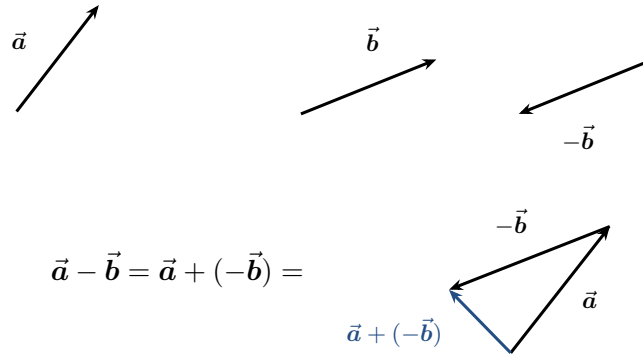
If the vector is divided by 3, its components are divided by 3.

4 Vector Subtraction

4.1 Method 1: add the negative vector

$$\vec{a} - \vec{b} = \vec{a} + (-\vec{b}).$$

Graphically, if $\vec{a} = (3, 4)$ and $\vec{b} = (5, 2)$, then



We can see that $\vec{a} - \vec{b} = (-2, 2)$.

Numerically,

$$\begin{aligned}\vec{a} - \vec{b} &= \vec{a} + (-\vec{b}) \\ &= (3, 4) + (-(5, 2)) \\ &= (3, 4) + (-5, -2) \\ &= (-2, 2).\end{aligned}$$

In other words,

$$\begin{aligned}\vec{a} - \vec{b} &= (3, 4) - (5, 2) \\ &= (3 - 5, 4 - 2) \\ &= (-2, 2).\end{aligned}$$

Subtract the corresponding components.

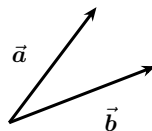
4.2 Method 2: tail-to-tail subtraction

Method 2 is more convenient when subtracting graphically. Let

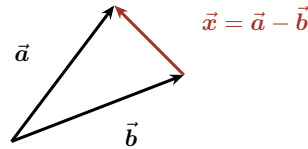
$$\vec{a} - \vec{b} = \vec{x}, \quad \text{so} \quad \vec{a} = \vec{b} + \vec{x}.$$

Thus $\vec{a} - \vec{b}$ (that is, $\vec{x}$) is the vector that must be added to $\vec{b}$ to get $\vec{a}$.

If we draw $\vec{a}$ and $\vec{b}$ tail-to-tail, this becomes easier to see:

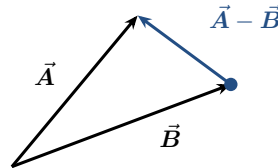


What do we add to $\vec{b}$ to get $\vec{a}$?



By following the vectors tip-to-tail, we can see that $\vec{b} + \vec{x} = \vec{a}$.

In general, to subtract vectors $\vec{A}$ and $\vec{B}$ graphically, place them tail-to-tail. Then $\vec{A} - \vec{B}$ goes from the tip of $\vec{B}$ to the tip of $\vec{A}$.



4.3 Worked examples

Example. Given $\vec{a} = (3, 4)$, $\vec{b} = (12, -18)$, and $\vec{c} = (-10, 14)$, find

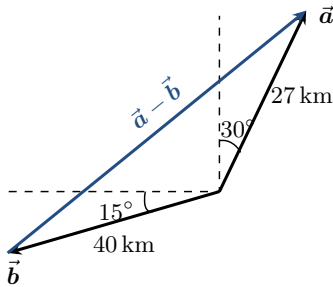
$$5\vec{a} - \frac{\vec{b}}{3} + 0.2\vec{c}.$$

$$\begin{aligned} 5\vec{a} - \frac{\vec{b}}{3} + 0.2\vec{c} &= 5(3, 4) - \frac{(12, -18)}{3} + 0.2(-10, 14) \\ &= (15, 20) - (4, -6) + (-2, 2.8) \\ &= (11, 26) + (-2, 2.8) \\ &= (9, 28.8). \end{aligned}$$

Example. Given

$$\vec{a} = 27 \text{ km } [30^\circ \text{ E of N}], \quad \vec{b} = 40 \text{ km } [15^\circ \text{ S of W}],$$

calculate $\vec{a} - \vec{b}$, giving the answer in polar form.



$$\begin{aligned} \vec{a} &= (27 \sin 30^\circ, 27 \cos 30^\circ) \\ &\simeq (13.5, 23.38), \end{aligned}$$

$$\begin{aligned} \vec{b} &= (-40 \cos 15^\circ, -40 \sin 15^\circ) \\ &\simeq (-38.64, -10.35). \end{aligned}$$

$$\begin{aligned} \vec{a} - \vec{b} &= (13.5, 23.38) - (-38.64, -10.35) \\ &= (52.14, 33.73), \end{aligned}$$

$$\|\vec{a} - \vec{b}\| = \sqrt{52.14^2 + 33.73^2} = 62.1,$$

$$\theta = \tan^{-1}\left(\frac{33.73}{52.14}\right) = 32.9^\circ.$$

Therefore

$$\vec{a} - \vec{b} = 62.1 \text{ km } [32.9^\circ \text{ N of E}].$$

5 Further Exercises

1. Recall that for airplanes,

$$\vec{v}_G = \vec{v}_a + \vec{v}_w.$$

If a pilot wants to fly at $800 \frac{\text{km}}{\text{h}}$ [37° N of E] relative to the ground, but the wind is blowing at $170 \frac{\text{km}}{\text{h}}$ *from the south*, in what direction should the pilot point the plane, and what should their airspeed be?

2. Using Method 2, draw $\vec{A} - \vec{B}$ for each pair:



3. Acceleration is change in velocity over change in time:

$$\vec{a} = \frac{\Delta \vec{v}}{\Delta t} = \frac{\vec{v}_2 - \vec{v}_1}{t_2 - t_1}.$$

At $t_1 = 3\text{ s}$, $\vec{v}_1 = (7 \frac{\text{m}}{\text{s}}, 15 \frac{\text{m}}{\text{s}})$, and at $t_2 = 4.3\text{ s}$, $\vec{v}_2 = (8.5 \frac{\text{m}}{\text{s}}, 18.1 \frac{\text{m}}{\text{s}})$. Calculate $\vec{a}$, the acceleration.

4. Conservation of momentum states that

$$m_1 \vec{v}_{1i} + m_2 \vec{v}_{2i} = m_1 \vec{v}_{1f} + m_2 \vec{v}_{2f}.$$

If $m_1 = 3\text{ kg}$ and $m_2 = 5\text{ kg}$, and the two objects are initially colliding with

$$\vec{v}_{1i} = \left(2 \frac{\text{m}}{\text{s}}, 3 \frac{\text{m}}{\text{s}}\right), \quad \vec{v}_{2i} = \left(-1.8 \frac{\text{m}}{\text{s}}, -1 \frac{\text{m}}{\text{s}}\right),$$

and the first object bounces off with $\vec{v}_{1f} = (-1 \frac{\text{m}}{\text{s}}, 1 \frac{\text{m}}{\text{s}})$, find $\vec{v}_{2f}$, the final velocity of the second object.

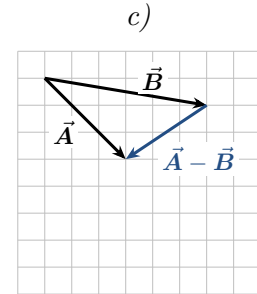
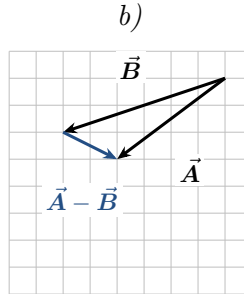
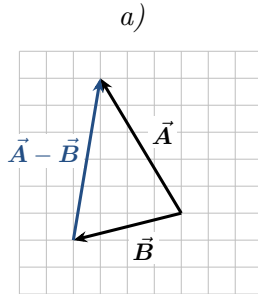
6 Answer Key: Further Exercises

1.

$$\begin{aligned}
 \vec{v}_a &= \vec{v}_G - \vec{v}_w \\
 &= [(800 \cos 37^\circ, 800 \sin 37^\circ) - (0, 170)] \frac{\text{km}}{\text{h}} \\
 &\simeq (638.91, 311.45) \frac{\text{km}}{\text{h}}.
 \end{aligned}$$

The pilot should point the plane 26.0° N of E, with an airspeed of $710.8 \frac{\text{km}}{\text{h}}$.

2.



3.

$$\begin{aligned}
 \vec{a} &= \frac{(8.5, 18.1) - (7, 15)}{4.3 - 3} \frac{\text{m}}{\text{s}^2} \\
 &= \left(\frac{15}{13}, \frac{31}{13} \right) \frac{\text{m}}{\text{s}^2} \\
 &\simeq \left(1.15 \frac{\text{m}}{\text{s}^2}, 2.38 \frac{\text{m}}{\text{s}^2} \right).
 \end{aligned}$$

4.

$$\begin{aligned}
 \vec{v}_{2f} &= \frac{3(2, 3) + 5(-1.8, -1) - 3(-1, 1)}{5} \frac{\text{m}}{\text{s}} \\
 &= \left(0, \frac{1}{5} \right) \frac{\text{m}}{\text{s}} = \left(0 \frac{\text{m}}{\text{s}}, 0.2 \frac{\text{m}}{\text{s}} \right).
 \end{aligned}$$